Pricing long-lived securities in dynamic endowment

Banothu Ganesh
AsstProfessor,Department of MBA,
Princeton College of Engineering and Tecnology,Ghatkesar,TS,India

Abstract

We solve for asset prices in a general affine representative-agent economy with isoelastic recursive utility and rare events. Our novel solution method is exact in two special cases: no preference for early resolution of uncertainty and elasticity of intertemporal substitution equal to one. Our results clarify model properties governed by the elasticity of intertemporal substitution, by risk aversion, and by the preference for early resolution of uncertainty. Our results also highlight that a covariance-based factor structure arises as a very special case, rather than as a general property, of equilibrium models.

Introduction

The framework of representative-agent asset pricing, in which complete markets allows for the diversification of idiosyncratic risks, has for many years delivered benchmark models of the cross-section and time-series of stock prices and returns.1 These models are at the same time simple and rich in the types of economic intuition they capture. In this paper, we focus on two nested sub-classes of the dynamic representative-agent framework with the goal of clarifying important implications for risk premia and asset prices. In the first part of the paper, we extend classic cross-sectional results of Merton (1973) and Breeden (1979) to a dynamic setting with recursive utility (Epstein and Zin, 1989) and rare events. This section assumes only isoelastic recursive utility and a Markov structure. Widening the class of models beyond the traditional diffusion framework of Merton (1973) has dramatic implications for the cross-section. The intertemporal capital asset pricing model (ICAPM) of Merton is a standard justifications for the near-universal use of covariance-based factor models in finance. However, the ICAPM relies on conditional log-normality. Without the assumption of conditional log-normality, a factor structure may not hold

In this section, we also show that the dynamics of the wealth-consumption ratio are principally governed by the elasticity of intertemporal substitution (EIS) and the risk premia in the cross-section relative to a consumption-based model are governed by the preference for early resolution of uncertainty. However, relative to a wealth-based model, risk premia in the cross-section are governed by risk aversion. When risk aversion is equal to one, a rare-event wealth CAPM holds regardless of the EIS. In contrast, a rare-event consumption CAPM holds if there is no preference for early resolution of uncertainty, regardless of risk aversion. Some of these comparative statics have appeared elsewhere in the literature, but the advantage of our frameowrk is that we can show them in a more general setting.

In the second part of the paper, we derive approximate analytical solutions for thepricing of long-lived assets. Our solution method takes as its starting point the widelyused method of Campbell and Shiller (1988) which involves a first-order approximation of the price-dividend ratio by a log-linear function. Previous studies use this method to compute the wealth-consumption ratio (which is necessary for computing other asset prices under recursive utility), and then to compute prices on other assets. While we use the method to compute the wealth-consumption ratio, we then, given the approximation, compute prices on other assets exactly. As a consequence, our method, unlike others, is exact both when the elasticity of intertemporal substitution is equal to one, and when utility is time-additive. The reason is that the approximation for the wealth-consumption ratio is exact when the EIS is equal to one, and a log-linear wealth-consumption ratio is not necessary for closed-form solutions for asset prices under time-additive utility

In an example, we extend the model of Wachter (2013) to a case of non-unitary EIS. We show that our technique is notably closer to the solution when the exact problem is solved numerically, than the standard approximation. Moreover, because our solution is closer to the true solution in a formal sense, it delivers insight into the economics of the problem.

The remainder of the paper proceeds as follows. Section 2 describes our general set-up and derives results for the cross-section. Section 3 describes the affine set-up with analytical solutions. Section 4 quantitatively evaluates the solution method under an example economy.

2.General Model

2.1 Assumptions Let Bct be a unidimensional Brownian motion and BXt an n-dimensional Brownian motion, such that Bct and BXt are independent. For j = 1, . . . , m, let Njt be independent Poisson processes. Consider functions μc : Rn → R, σc : Rn → R, μX : Rn → Rn , σX : Rn → Rn ,and λ : Rn → Rm. Assume the endowment follows the process

$$\frac{dC_t}{C_{t-}} = \mu_c(X_{t-})\,dt + \sigma_c(X_{t-})dB_{ct} + \sum_{j=1}^{m}(e^{Z_{cjt}} - 1)dN_{jt}, \tag{1}$$

where X_t is a vector of state variables following the process

$$dX_t = \mu_X(X_{t-})\,dt + \sigma_X(X_{t-})dB_{Xt} + \sum_{j=1}^{m} Z_{Xjt}dN_{jt}, \tag{2}$$

and where, for all j = 1, . . . , m and t, Zcjt are scalar random variables and ZXjt an n × 1 vector of random variables. We assume the joint distribution of Zcjt and ZXjt is timeinvariant, and thus suppress the t subscript when not essential for clarity. The intensity for Poisson process Njt is time-varying and given by λj (x), the j-th element of λ(x). We adopt the convention that BXt, and therefore Xt , are column vectors and that σc is a row vector.

Consider an asset that pays cash flows determined by the outcome of a dividend process dDt Dt− = μd(Xt) + σd(Xt)dBct + Xm j=1 e Zdjt − 1 dNjt. (3)

There may be many such assets, but because we will assume complete markets, it will suffice to consider each asset in isolation, and therefore it is not necessary to add a subscript to Dt . Similarly to the rare-event outcomes for consumption and for Xt ,Zdjt is a random variable with time-invariant distribution for all j. 2 In what follows, let Bt = [Bct, B>Xt] >. For a generic function h(Ct , Dt , Xt), definewhereEvj denotes expectations taken with respect to the joint distribution of Zcj , Zdj , ZXj , and conditional on information just prior to t. Assume a representative agent with recursively-defined utilitywhere ψ is the elasticity of intertemporal substitution and γ is risk aversion (Duffie and Epstein, 1992b). When γ = 1/ψ, the recursion in (5) is linear, and (4) reduces to the time-additive case. We are interested in two limiting cases of (5). When ψ = 1, (5) reduces to

2.2 General characterization of the solution

We first characterize the value function, the wealth-consumption ratio, and the state-price density in terms of the state variables. Here and in the remainder of the paper, we follow the convention that partial derivatives with respect to a vector are row vectors; for example, ∂I/∂X = [∂I/∂X1, · · · , ∂I/∂Xn]. Proofs not given in the main text are contained in Appendix A. Proposition 1 (Value function). Suppose the representative agent's preference is defined by (4)–(6), where the consumption growth process follows (1) and the state variable process follows (2). In equilibrium, J(Ct , Xt) = Vt , where

$$J(C_t, X_t) = \frac{C_t^{1-\gamma} I(X_t)^{1-\gamma}}{1-\gamma}, \tag{8}$$

for $\gamma \neq 1$ and

$$J(C_t, X_t) = \log C_t + \log I(X_t)$$

for $\gamma = 1$, where $I(\cdot)$ satisfies the partial differential equation

$$\frac{\beta}{1-\frac{1}{\psi}}\left[I(x)^{\frac{1}{\psi}-1} - 1\right] + \mu_c(x) + \frac{1}{2}\mathrm{tr}\left[\left(\frac{1}{I}\frac{\partial^2 I}{\partial x^2} - \frac{\gamma}{I^2}\left(\frac{\partial I}{\partial x}\right)^{\top}\left(\frac{\partial I}{\partial x}\right)\right)\sigma(x)\sigma(x)^{\top}\right]$$
$$+ \frac{1}{I}\frac{\partial I}{\partial x}\mu_X(x) - \frac{1}{2}\gamma\sigma_c(x)^2 + \frac{1}{1-\gamma}\sum_{j=1}^{m}\lambda_j(x)E_{\nu_j}\left[e^{(1-\gamma)Z_{cj}}\left(\frac{I(x+Z_{Xj})}{I(x)}\right)^{1-\gamma} - 1\right] = 0. \tag{9}$$

for $\psi \neq 1$ and

$$-\beta\log I + \mu_c(x) + \frac{1}{2}\mathrm{tr}\left[\left(\frac{1}{I}\frac{\partial^2 I}{\partial x^2} - \frac{\gamma}{I^2}\left(\frac{\partial I}{\partial x}\right)^{\top}\left(\frac{\partial I}{\partial x}\right)\right)\sigma(x)\sigma(x)^{\top}\right]$$
$$+ \frac{1}{I}\frac{\partial I}{\partial x}\mu_X(x) - \frac{1}{2}\gamma\sigma_c(x)^2 + \frac{1}{1-\gamma}\sum_{j=1}^{m}\lambda_j(x)E_{\nu_j}\left[e^{(1-\gamma)Z_{cj}}\left(\frac{I(x+Z_{Xj})}{I(x)}\right)^{1-\gamma} - 1\right] = 0. \tag{10}$$

We solve for asset prices in a general affine representative-agent economy with isoelastic recursive utility and rare events. Our novel solution method is exact in two special cases: no preference for early resolution of uncertainty and elasticity of intertemporal substitution equal to one. Our results clarify model properties governed by the elasticity of intertemporal substitution, by risk aversion, and by the preference for early resolution of uncertainty. Our results also highlight that a covariance-based factor structure arises as a very special case, rather than as a general property, of equilibrium models

for ψ = 1. Equation 10 is a special case of (9) as can be seen by taking the limit as ψ → 1. Given the value function (8), we can now express the wealth-consumption ratio and the state price density in terms of I(Xt). The consumption-wealth ratio will play an important role in our solution method for the affine case. Corollary 2 (Wealth-consumption ratio). Let Wt denote the wealth of the representative agent at time t. Then the wealth-to-consumption ratio Gc (Xt) ≡ Wt/Ct is a function of Xt and is given by

$$G^c(X_t) = \begin{cases} \beta^{-1} I(X_t)^{1-\frac{1}{\psi}} & \psi \neq 1 \\ \beta^{-1} & \psi = 1. \end{cases} \tag{11}$$

Conclusions

The second and third terms on the right-hand side of (A.29) have zero expectation. Therefore the first term in (A.29) must also have zero expectation, and it follows that the integrand of this term must equal zero.as can be seen by taking the limit as ψ → 1. Given the value function (8), we can now express the wealth-consumption ratio and the state price density in terms of I(Xt). The consumption-wealth ratio will play an important role in our solution method for the affine case. Corollary 2 (Wealth-consumption ratio). Let Wt denote the wealth of the representative agent at time t. Then the wealth-to-consumption ratio is usefull

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